Time Varying Hierarchical Archimedean Copulae (*HALOC*)

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Simple AC over time

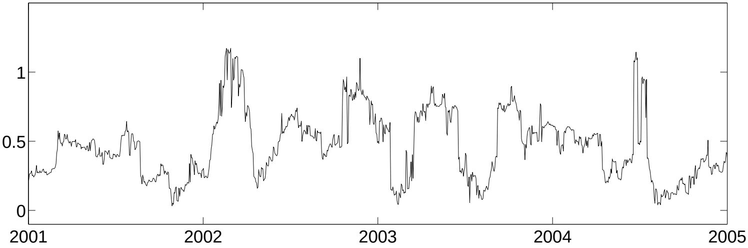
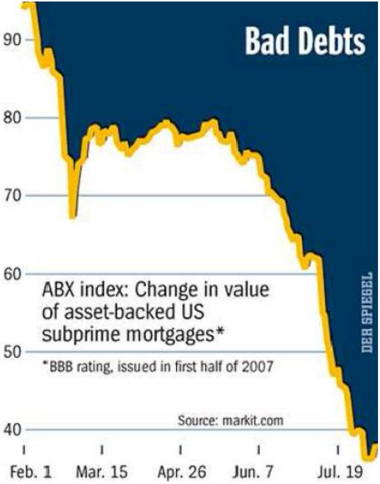


Figure 1: Estimated copula dependence parameter $\widehat{\theta}_t$ with the Local Change Point method for 6-dimensional data: DC, VW, Bayer, BASF, Allianz and Münchener Rückversicherung. Clayton Copula.
Giacomini et. al (2009)

Collateralized Debt Obligation

Triggered the financial crisis.



CDO Dynamics

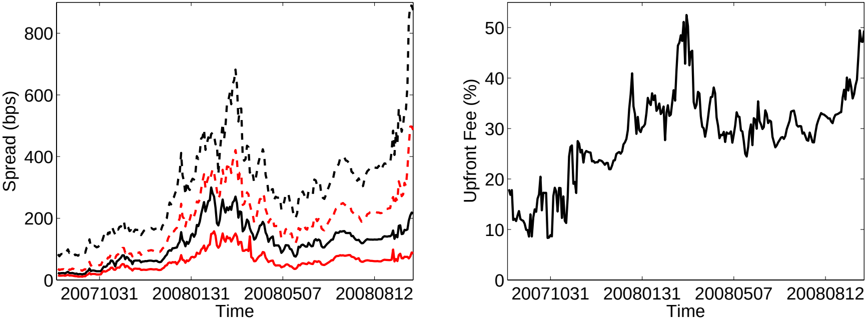


Figure 2: Spreads of iTraxx tranches, Series 8, maturity 5 years, data from 20070920-20081022. Left panel: mezzanine junior (dashed black), mezzanine (dashed red), senior (solid black), super senior (solid red). Right panel: upfront fee of the equity tranche.

Dependence Matters!

The normal world is not enough.

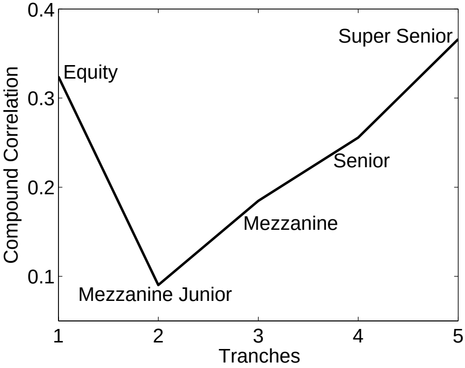


Figure 3: Gaussian one factor model with constant correlation. Data from 20071022.

Time Varying Structures

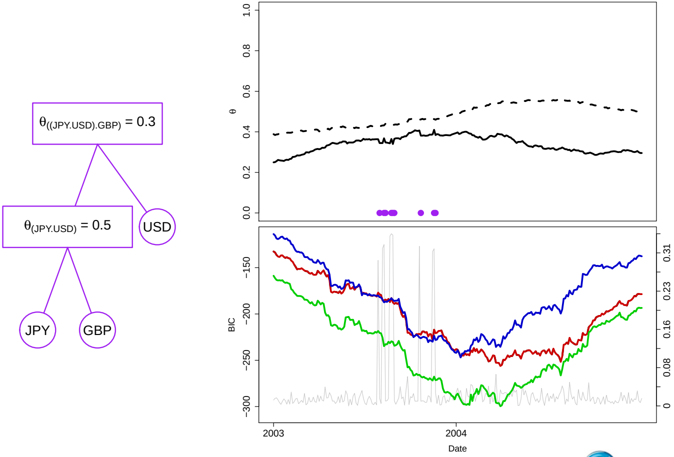


Figure 4: Film of changing structures over time.

Main Idea

- combine interpretability with flexibility of copulae
- determine the structure of HAC for a given time series
- identify time varying dependencies
- apply to risk pattern analysis
- reduce dimension of dependency

Outline

1. Motivation ✓
2. Hierarchical Archimedean copulae
3. Local Parametric Modeling by HAC
4. Simulation Study
5. Empirical Part
6. References

Copula

For a distribution function F with marginals $F_{X_1} \ldots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \tag{1}$$

for all $x_i \in \overline{\mathbb{R}}$, $i = 1, \ldots, d$. If $F_{X_1}, \ldots, F_{X_d}$ are cts, then C is unique. If C is a copula and $F_{X_1}, \ldots, F_{X_d}$ are cdfs, then the function F defined in (1) is a joint cdf with marginals $F_{X_1}, \ldots, F_{X_d}$.

A little bit of history

- 1940s: *Wassilij Hoeffding* studies properties of multivariate distributions

1914–91, b. Mustamäki, Finland; d. Chapel Hill, NC
gained his PhD from U Berlin in 1940
1924–45 work in U Berlin

A little bit of history

- 1940s: *Wassilij Hoeffding* studies properties of multivariate distributions
- 1959: The word **copula** appears for the first time (*Abe Sklar*)
- 1999: Introduced to financial applications (*Paul Embrechts*, *Alexander McNeil*, *Daniel Straumann* in RISK Magazine)
- 2000: Paper by *David Li* in *Journal of Derivatives* on application of copulae to CDO
- 2006: Several insurance companies, banks and other financial institutions apply copulae as a risk management tool

Elliptical Gaussian Copula

$$C_\delta^G(u_1, u_2) = \boldsymbol{\Phi}_\delta\{\boldsymbol{\Phi}^{-1}(u_1), \boldsymbol{\Phi}^{-1}(u_2)\}$$

$$= \int_{-\infty}^{\boldsymbol{\Phi}^{-1}(u_1)} \int_{-\infty}^{\boldsymbol{\Phi}^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\delta^2}} \exp\left\{\frac{-(s^2 - 2\delta st + t^2)}{2(1-\delta^2)}\right\} ds\, dt,$$

- Gaussian copula contains the dependence structure
- *normal* marginal distribution + Gaussian copula = multivariate normal distributions
- *non-normal* marginal distribution + Gaussian copula = meta-Gaussian distributions
- allows to generate joint symmetric dependence, but no tail dependence

Archimedean Copulae

Multivariate Archimedean copula $C : [0,1]^d \to [0,1]$ defined as

$$C(u_1, \ldots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \qquad (2)$$

where $\phi : [0, \infty) \to [0,1]$ is continuous and strictly decreasing with $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} its pseudo-inverse.

Example

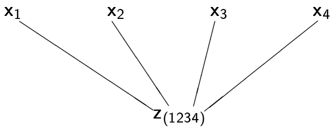
$$\begin{aligned} \phi_{Gumbel}(u, \theta) &= \exp\{-u^{1/\theta}\}, \text{ where } 1 \leq \theta < \infty \\ \phi_{Clayton}(u, \theta) &= (\theta u + 1)^{-1/\theta}, \text{ where } \theta \in [-1, \infty)\backslash\{0\} \end{aligned}$$

Disadvantages: too restrictive, single parameter, exchangeable

Hierarchical Archimedean Copulae

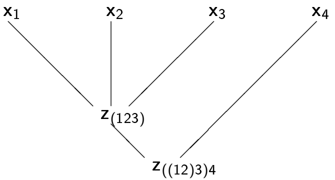
Simple AC with s=(1234)

$$C(u_1, u_2, u_3, u_4) = C_1(u_1, u_2, u_3, u_4)$$



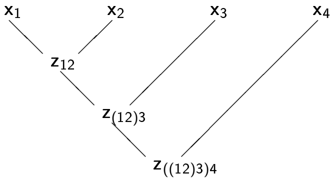
AC with s=((123)4)

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2, u_3), u_4\}$$



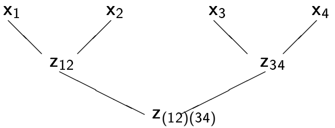
Fully nested AC with s=(((12)3)4)

$$C(u_1, u_2, u_3, u_4) = C_1[C_2\{C_3(u_1, u_2), u_3\}, u_4]$$



Partially Nested AC with s=((12)(34))

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2), C_3(u_3, u_4)\}$$



Hierarchical Archimedean Copulae

Advantages of HAC:

- flexibility and wide range of dependencies:
  for $d = 10$ more than $2.8 \cdot 10^8$ structures
- dimension reduction:
  $d - 1$ parameters to be estimated
- subcopulae are also HAC

Hierarchical Archimedean Copulae

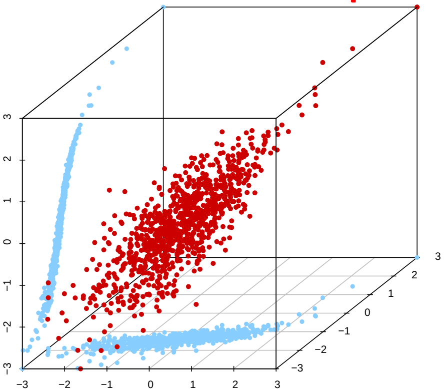


Figure 5: Scatterplot of the $C_{Gu}[C_{Gu}\{\Phi(x_1), t_2(x_2); \theta(\tau_1) = \theta(0.5) = 2\}, \Phi(x_3); \theta(\tau_2) = \theta(0.9) = 10]$

Hierarchical Archimedean Copulae

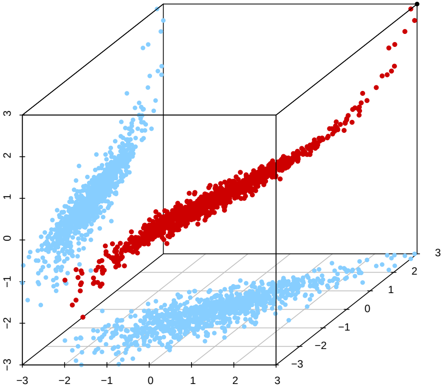


Figure 6: Scatterplot of the $C_{Gu}[\Phi(x_2), C_{Gu}\{t_2(x_1), \Phi(x_3); \theta(\tau_1) = \theta(0.5) = 2\}; \theta(\tau_2) = \theta(0.9) = 10]$

Determining Structure

$(12) \rightsquigarrow \widehat{\theta}_{12}$

$(13) \rightsquigarrow \widehat{\theta}_{13}$

$(14) \rightsquigarrow \widehat{\theta}_{14}$

$(23) \rightsquigarrow \widehat{\theta}_{23}$

$(24) \rightsquigarrow \widehat{\theta}_{24}$

$(34) \rightsquigarrow \widehat{\theta}_{34}$

$(123) \rightsquigarrow \widehat{\theta}_{123}$

$(124) \rightsquigarrow \widehat{\theta}_{124}$

$(234) \rightsquigarrow \widehat{\theta}_{234}$

$(134) \rightsquigarrow \widehat{\theta}_{134}$

$(1234) \rightsquigarrow \widehat{\theta}_{1234}$

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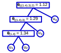
$(234) \rightsquigarrow \widehat{\theta}_{234}$
$(134) \rightsquigarrow \widehat{\theta}_{134}$
$(1234) \rightsquigarrow \widehat{\theta}_{1234}$

best fit (13) $\rightsquigarrow$

$$z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}$$



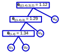
Determining Structure

$$
\begin{array}{c}
(12) \rightsquigarrow \widehat{\theta}_{12} \\
\color{red}{(13)} \rightsquigarrow \widehat{\theta}_{13} \\
(14) \rightsquigarrow \widehat{\theta}_{14} \\
(23) \rightsquigarrow \widehat{\theta}_{23} \\
(24) \rightsquigarrow \widehat{\theta}_{24} \\
(34) \rightsquigarrow \widehat{\theta}_{34} \\
\hline
(123) \rightsquigarrow \widehat{\theta}_{123} \\
(124) \rightsquigarrow \widehat{\theta}_{124} \\
(234) \rightsquigarrow \widehat{\theta}_{234} \\
(134) \rightsquigarrow \widehat{\theta}_{134} \\
(1234) \rightsquigarrow \widehat{\theta}_{1234}
\end{array}
$$

best fit (13) $\rightsquigarrow$

$$
\boxed{z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}}
$$

$$
\begin{array}{c}
(13)2 \rightsquigarrow \widehat{\theta}_{(13)2} \\
(13)4 \rightsquigarrow \widehat{\theta}_{(13)4} \\
24 \rightsquigarrow \widehat{\theta}_{24} \\
\hline
(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}
\end{array}
$$



Determining Structure

$(12) \rightsquigarrow \widehat{\theta}_{12}$
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$(1234) \rightsquigarrow \widehat{\theta}_{1234}$

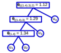
best fit (13) $\rightsquigarrow$

$$z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}$$

$(13)2 \rightsquigarrow \widehat{\theta}_{(13)2}$
$(13)4 \rightsquigarrow \widehat{\theta}_{(13)4}$
$24 \rightsquigarrow \widehat{\theta}_{24}$
$(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}$

best fit ((13)4) $\rightsquigarrow$

$$z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}$$



Determining Structure

$$
\left.\begin{array}{c}
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(14) \rightsquigarrow \widehat{\theta}_{14}\\
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(134) \rightsquigarrow \widehat{\theta}_{134}\\
(1234) \rightsquigarrow \widehat{\theta}_{1234}
\end{array}\right| \text{best fit } (13) \rightsquigarrow
$$

$$
\boxed{z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}}
$$

$$
\left.\begin{array}{c}
(13)2 \rightsquigarrow \widehat{\theta}_{(13)2}\\
(13)4 \rightsquigarrow \widehat{\theta}_{(13)4}\\
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\hline
(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}
\end{array}\right| \text{best fit } ((13)4) \rightsquigarrow
$$

$$
\boxed{z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}}
$$

$$
((13)4)2 \rightsquigarrow \widehat{\theta}_{((13)4)2}
$$

Determining Structure

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$(1234) \rightsquigarrow \widehat{\theta}_{1234}$

best fit (13) $\rightsquigarrow$

$$z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}$$

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$(13)4 \rightsquigarrow \widehat{\theta}_{(13)4}$
$24 \rightsquigarrow \widehat{\theta}_{24}$
$(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}$

best fit ((13)4) $\rightsquigarrow$

$$z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}$$

$((13)4)2 \rightsquigarrow \widehat{\theta}_{((13)4)2}$

Estimation: multistage MLE with nonparametric and parametric margins

Criteria for grouping: goodness-of-fit tests, parameter-based method, etc.

Criteria for grouping

- goodness-of-fit tests
  - dimension dependent
  - KS type tests (difficult)
  - probability integral transformation
- Ali-Silvey distance measures
  - dimension dependent

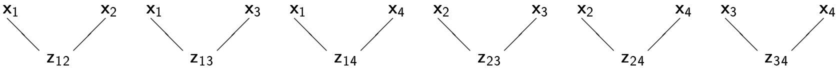
Criteria for grouping

- parameter-based methods
  Note that, if the true structure is (123) then

$$\theta_{(12)} = \theta_{(13)} = \theta_{(23)} \quad = \theta_{(123)}$$

  - heuristic methods - based on proximity between parameters on different levels
  - test-based methods - based on tests for the parameters
- tests on exchangeability

Criteria for grouping based on θ's



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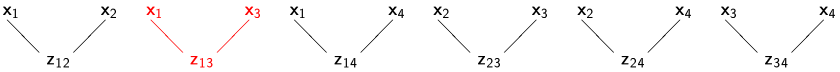
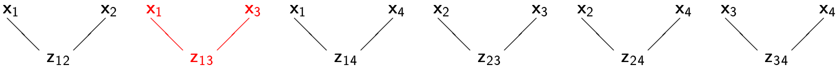


$$\max\{\widehat{\theta}_{12}, \widehat{\theta}_{13}, \widehat{\theta}_{14}, \widehat{\theta}_{23}, \widehat{\theta}_{24}, \widehat{\theta}_{34}\} = \widehat{\theta}_{13} \quad \Rightarrow$$

Criteria for grouping based on θ's



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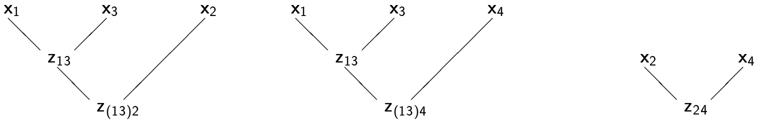

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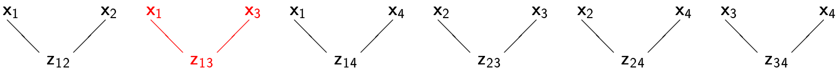

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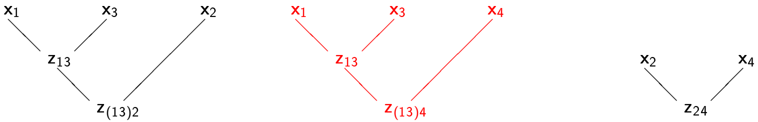
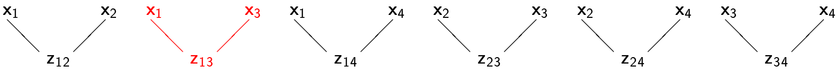


$$\max\{\widehat{\theta}_{(13)2}, \widehat{\theta}_{(13)4}, \widehat{\theta}_{24}\} = \widehat{\theta}_{(13)4} \quad \Rightarrow$$

Criteria for grouping based on θ's



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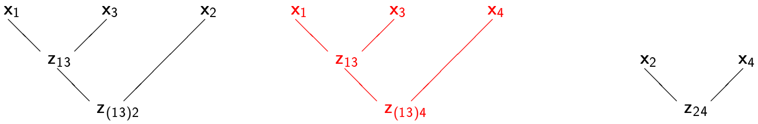

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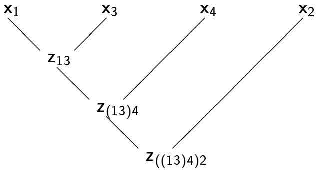

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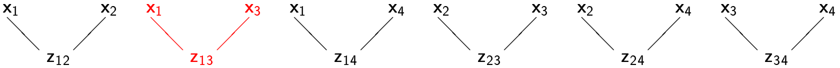

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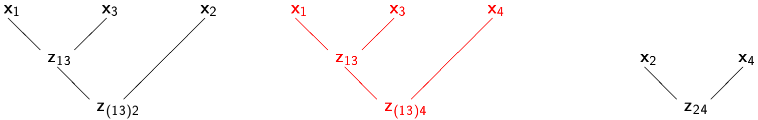

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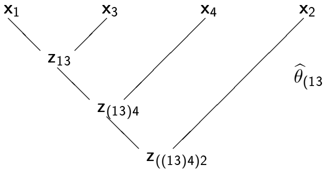
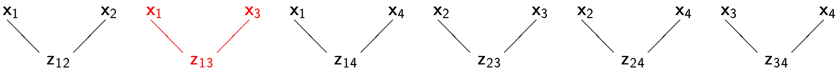


$$\widehat{\theta}_{(13)} \approx \widehat{\theta}_{((13)4)} \Rightarrow$$

Criteria for grouping based on θ's



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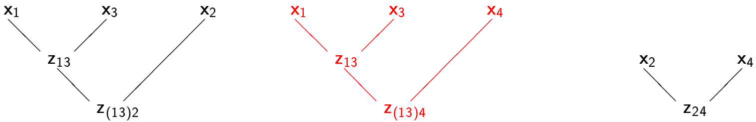

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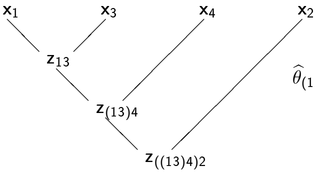
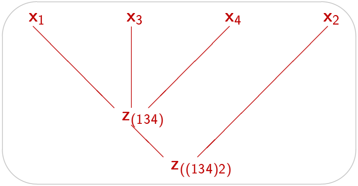


$\widehat{\theta}_{(13)} \approx \widehat{\theta}_{((13)4)} \Rightarrow$



Local Change Point Detection

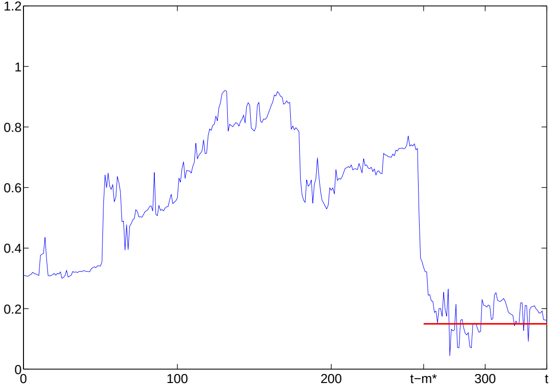


Figure 7: Dependence over time for DaimlerChrysler, Volkswagen, Bayer, BASF, Allianz and Münchener Rückversicherung, 20000101-20041231. Giacomini et. al (2009)

Adaptive Copula Estimation

- adaptively estimate largest interval where homogeneity hypothesis is accepted
- *Local Change Point* detection (LCP): sequentially test $\theta_t,\ s_t$ are constants (i.e. $\theta_t = \theta,\ s_t = s$) within some interval I (local parametric assumption).

- "*Oracle*" choice: largest interval $I = [t_0 - m_{k^*}, t_0]$ where small modelling bias condition (SMB)

$$\triangle_I(s, \boldsymbol{\theta}) = \sum_{t \in I} \mathcal{K}\{C(\cdot; s_t, \boldsymbol{\theta}_t), C(\cdot; s, \boldsymbol{\theta})\} \leq \triangle.$$

holds for some $\triangle \geq 0$. m_{k^*} is the ideal scale, $(s, \ \theta)^\top$ is ideally estimated from $I = [t_0 - m_{k^*}, t_0]$ and $\mathcal{K}(\cdot, \cdot)$ is the *Kullback-Leibler* divergence

$$\mathcal{K}\{C(\cdot; s_t, \boldsymbol{\theta}_t), C(\cdot; s, \boldsymbol{\theta})\} = \boldsymbol{E}_{s_t, \boldsymbol{\theta}_t} \log \frac{c(\cdot; s_t, \boldsymbol{\theta}_t)}{c(\cdot; s, \boldsymbol{\theta})}$$

Under the SMB condition on I_{k^*} and assuming that $\max_{k \le k^*} \boldsymbol{E}_{s_t, \boldsymbol{\theta}_t} |\mathcal{L}(\widetilde{s}_k, \widetilde{\boldsymbol{\theta}}_k) - \mathcal{L}(s, \boldsymbol{\theta})|^r \le \mathcal{R}_r(s_t, \boldsymbol{\theta}_t)$, we obtain

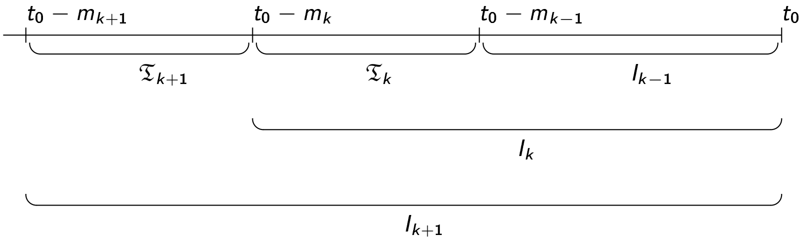
$$\boldsymbol{E}_{s_t, \boldsymbol{\theta}_t} \log \left\{ 1 + \frac{|\mathcal{L}(\widetilde{s}_{\widehat{k}}, \widetilde{\boldsymbol{\theta}}_{\widehat{k}}) - \mathcal{L}(s, \boldsymbol{\theta})|^r}{\mathcal{R}_r(s, \boldsymbol{\theta})} \right\} \le 1 + \triangle,$$

$$\boldsymbol{E}_{s_t, \boldsymbol{\theta}_t} \log \left\{ 1 + \frac{|\mathcal{L}(\widetilde{s}_{\widehat{k}}, \widetilde{\boldsymbol{\theta}}_{\widehat{k}}) - \mathcal{L}(\widehat{s}_{\widehat{k}}, \widehat{\boldsymbol{\theta}}_{\widehat{k}})|^r}{\mathcal{R}_r(s, \boldsymbol{\theta})} \right\} \le 1 + \triangle,$$

where $\widehat{a}_I$ is an adaptive estimator on I and $\widetilde{a}_I$ is any other parametric estimator on I.

Local Change Point Detection

1. define family of nested intervals
   $I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_K = I_{K+1}$ with length m_k as
   $I_k = [t_0 - m_k, t_0]$
2. define $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$



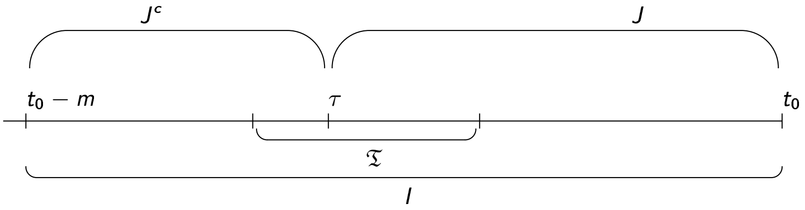
Local Change Point Detection

1. test homogeneity $H_{0,k}$ against the change point alternative in $\mathfrak{T}_k$ using I_{k+1}
2. if no change points in $\mathfrak{T}_k$, accept I_k. Take $\mathfrak{T}_{k+1}$ and repeat previous step until $H_{0,k}$ is rejected or largest possible interval I_K is accepted
3. if $H_{0,k}$ is rejected in $\mathfrak{T}_k$, homogeneity interval is the last accepted $\widehat{I} = I_{k-1}$
4. if largest possible interval I_K is accepted $\widehat{I} = I_K$

Test of homogeneity

Interval $I = [t_0 - m, t_0], \mathfrak{T} \subset I$

$$
\begin{aligned}
H_0 \quad &: \quad \forall \tau \in \mathfrak{T},\ \theta_t = \theta,\ s_t = s, \\
&\quad\ \ \forall t \in J = [\tau, t_0], \forall t \in J^c = I \setminus J \\
H_1 \quad &: \quad \exists \tau \in \mathfrak{T},\ \theta_t = \theta_1,\ s_t = s_1;\ \forall t \in J, \\
&\quad\ \ \theta_t = \theta_2 \neq \theta_1;\ s_t = s_2 \neq s_1, \forall t \in J^c
\end{aligned}
$$



Test of homogeneity

Likelihood ratio test statistic for fixed change point location:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2}\{L_J(\theta_1) + L_{J^c}(\theta_2)\} - \max_{\theta} L_I(\theta) \\ &= ML_J + ML_{J^c} - ML_I \end{aligned}$$

Test statistic for unknown change point location:

$$T_I = \max_{\tau \in \mathfrak{T}_I} T_{I,\tau}$$

Reject H_0 if for a critical value ζ_I

$$T_I > \zeta_I$$

Selection of I_k and ζ_k

- set of numbers m_k defining the length of I_k and $\mathfrak{I}_k$ are in the form of a geometric grid
- $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K$, $m_0 \in \{20,\ 40\}$, $c = 1.25$ and $K = 10$, where $[x]$ means the integer part of x
- $I_k = [t_0 - m_k, t_0]$ and $\mathfrak{I}_k = [t_0 - m_k, t_0 - m_{k-1}]$ for $k = 1, 2, \ldots, K$

(Mystery Parameters)

Sequential choice of ζ_k

- after k steps are two cases: change point at some step $\ell \le k$ and no change points.
- let $\mathcal{B}_\ell$ be the event meaning the rejection at step ℓ

$$\mathcal{B}_\ell = \{ T_1 \le \zeta_1, \ldots, T_{\ell-1} \le \zeta_{\ell-1}, T_\ell > \zeta_\ell \},$$

and $(\widehat{s}_k, \widehat{\boldsymbol{\theta}}_k) = (\widetilde{s}_{\ell-1}, \widetilde{\boldsymbol{\theta}}_{\ell-1})$ on $\mathcal{B}_\ell$ for $\ell = 1, \ldots, k$.

- we find sequentially such a minimal value of ζ_ℓ that ensures following inequality

$$\max_{k=l,\ldots,K} \boldsymbol{E}_{s^*,\theta^*} |\mathcal{L}(\widetilde{s}_k, \widetilde{\boldsymbol{\theta}}_k) - \mathcal{L}(\widetilde{s}_{\ell-1}, \widetilde{\boldsymbol{\theta}}_{\ell-1})|^r \mathbf{I}(\mathcal{B}_\ell) \le \rho \mathcal{R}_r(s^*, \boldsymbol{\theta}^*) k/K$$

Sequential choice of ζ_k

1. pairs of Kendall's τ: $\forall\{\tau_1, \tau_2\} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}^2, \ \tau_1 \geq \tau_2$
2. simul. from $C_{\theta_i,\theta_j}(u_1, u_2, u_3) = C\{C(u_1, u_2; \theta_1), u_3; \theta_2\}, \ \theta = \theta(\tau)$
3. run sequential algorithm for each sample

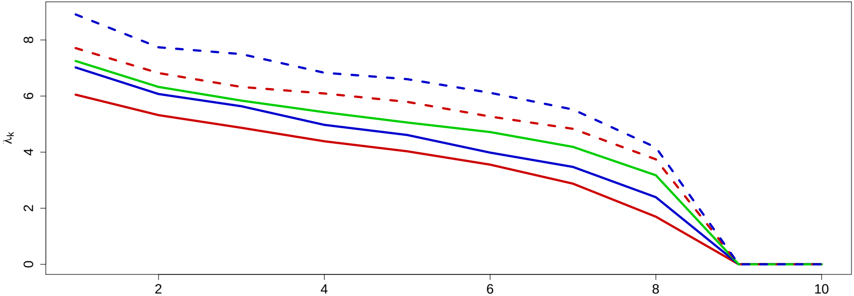


Figure 8: ζ_k of the 3-dimensional HAC as a function of k with the fixed $m_0 = 40$, $\rho = 0.5$, $r = 0.5$, $\tau_1 = 0.1$ and for different τ_2. $\tau_2 = 0.1$ (solid), $\tau_2 = 0.3$ (solid), $\tau_2 = 0.5$ (solid), $\tau_2 = 0.7$ (dashed), $\tau_2 = 0.9$ (dashed).

Simulation: Change in θ_1, I

$$C_t(u_1,u_2,u_3;s,\boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2,u_3;\theta_1 = 3.33);\theta_2 = 1.43\} & \text{for } 1 \le t \le 200 \\ C\{u_1, C(u_2,u_3;\theta_1 = 2.00);\theta_2 = 1.43\} & \text{for } 200 < t \le 400 \end{cases}$$

1. $N = 400$ and 100 runs
2. LCP based on the same critical values

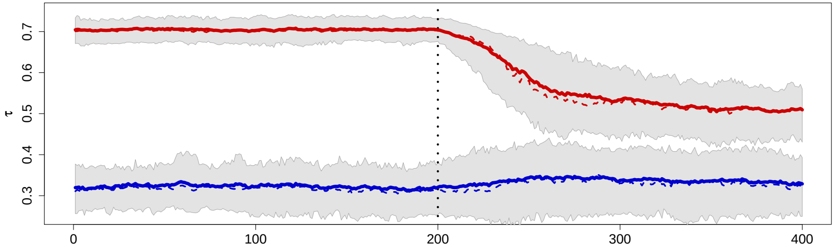


Figure 9: θ_1 and θ_2 on the intervals of homogeneity (median - dashed line, mean - solid line).

Simulation: Change in θ_1, II

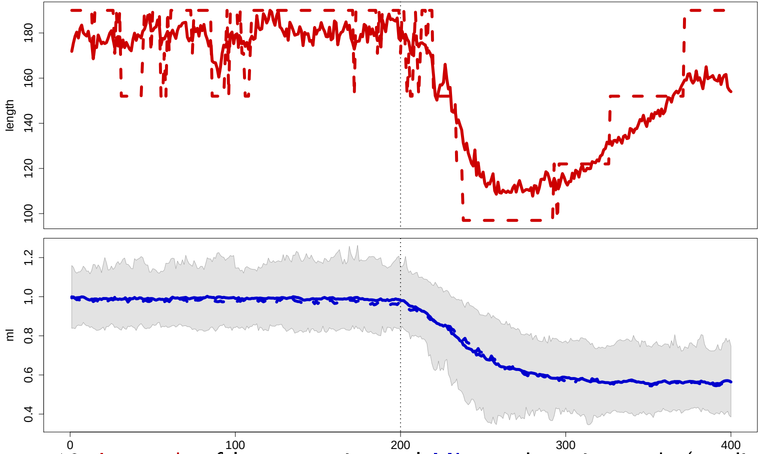


Figure 10: Intervals of homogeneity and ML on these intervals (median - dashed line, mean - solid line)

Simulation: Change in θ_2, I

$$C_t(u_1, u_2, u_3; s, \boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2, u_3; \theta_1 = 3.33); \theta_2 = 1.43\} & \text{for } 1 \le t \le 200 \\ C\{u_1, C(u_2, u_3; \theta_1 = 3.33); \theta_2 = 2.00\} & \text{for } 200 < t \le 400 \end{cases}$$

1. $N = 400$ and 100 runs
2. LCP based on the same critical values

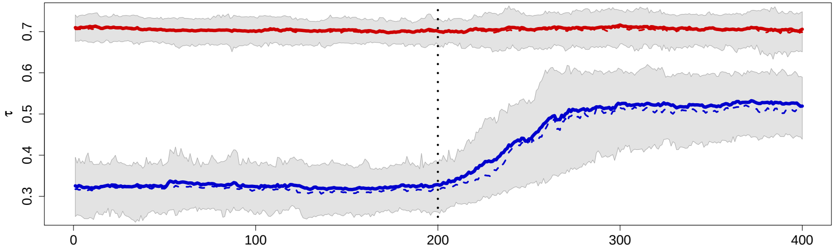


Figure 11: θ_1 and θ_2 on the intervals of homogeneity (median - dashed line, mean - solid line).

Simulation: Change in θ_2, II

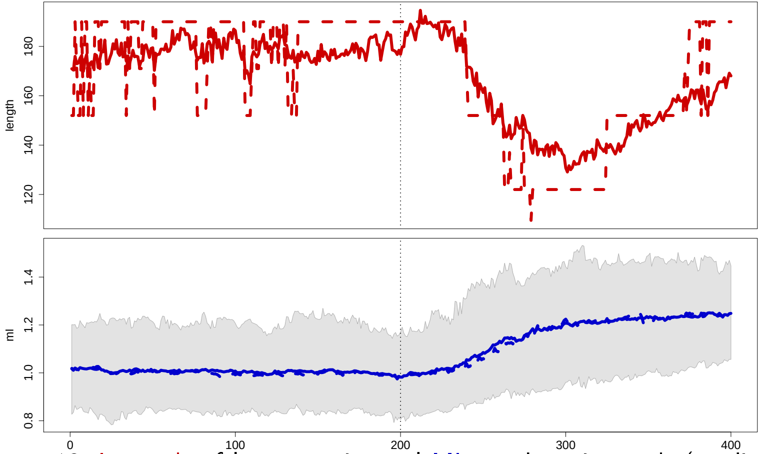


Figure 12: Intervals of homogeneity and ML on these intervals (median - dashed line, mean - solid line)

Simulation: Change in the Structure, I

$$C_t(u_1, u_2, u_3; s, \boldsymbol{\theta}) = \begin{cases} C\{u_1, C(u_2, u_3; \theta_1 = 3.33); \theta_2 = 1.43\} & \text{for } 1 \leq t \leq 200 \\ C\{C(u_1, u_2; \theta_1 = 3.33), u_3; \theta_2 = 1.43\} & \text{for } 200 < t \leq 400 \end{cases}$$

1. $N = 400$ and 100 runs
2. LCP based on the same critical values

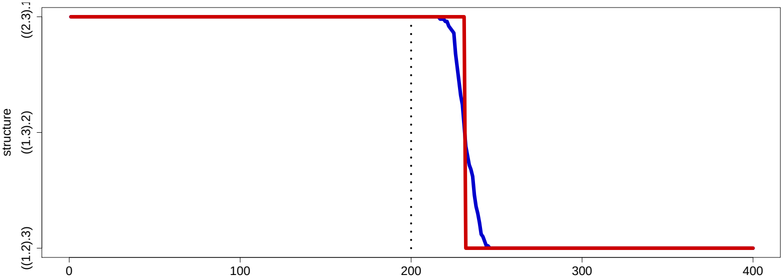


Figure 13: The structure of the est. HAC on the intervals of homogeneity (median - dashed line, mean - solid line)

Simulation: Change in the Structure, II

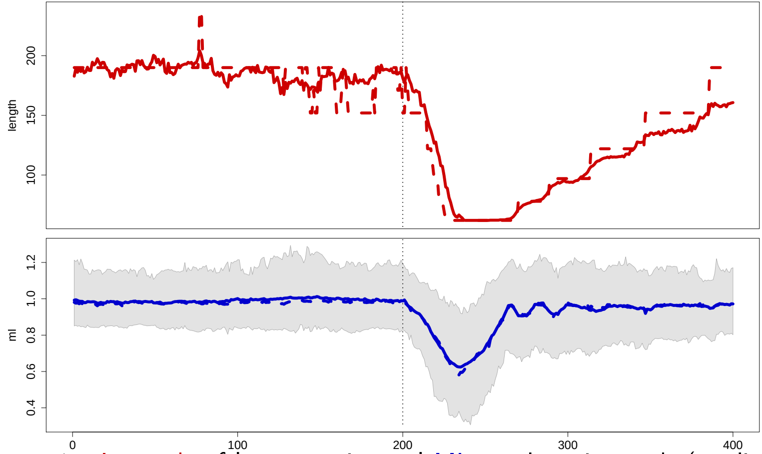


Figure 14: Intervals of homogeneity and ML on these intervals (median - dashed line, mean - solid line)

Data and Copula

daily values for the exchange rates
JPN/EUR, GBP/EUR and USD/EUR

timespan = [4.1.1999; 14.8.2009] ($n = 2771$)

$\mathcal{M} = \{\phi = \exp(-u^{1/\theta})\}$ - Gumbel generator

Data and Copula

- a univariate GARCH(1,1) process on log-returns

$$
\begin{aligned}
X_{j,t} &= \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t} \text{ with } \sigma^2_{j,t} = \omega_j + \alpha_j \sigma^2_{j,t-1} + \beta_j (X_{j,t-1} - \mu_{j,t-1})^2 \\
\varepsilon_t &\sim C\{F_1(x_1), \dots, F_d(x_d); \theta_t\}
\end{aligned}
$$

- estimated copula from the whole sample

$$s^* = (\text{JPY USD})_{1.588}\ \text{GBP}_{1.418}$$

	$\widehat{\mu}_j$	$\widehat{\omega}_j$	$\widehat{\alpha}_j$	$\widehat{\beta}_j$	BL	KS
JPY	4.85e-05	2.99e-07	0.06	0.94	0.73	1.70e-05
	(1.15e-04)	(1.02e-07)	(7.49e-03)	(7.64e-03)		
GBP	6.34e-05	1.44e-07	0.06	0.93	0.01	2.10e-04
	(7.39e-05)	(5.11e-08)	(8.75e-03)	(9.12e-03)		
USD	1.76e-04	1.19e-07	0.03	0.97	0.87	1.65e-03
	(1.10e-04)	(5.92e-08)	(4.14e-03)	(4.28e-03)		

Table 1: Estimation results univariate time series modelling.

Rolling window

$$ML = \sum_{i=1}^{n} \log\{f(u_{i1}, \dots, u_{id}, \widehat{\theta})\},$$

where f denotes the joint multivariate density function.

$$AIC = -2ML + 2m, \qquad BIC = -2ML + 2\log(m),$$

where m is the number of parameters to be estimated.
$\mathbf{\Theta}_t(d \times d)$ - matrix of the pairwise θ based on the 250 days before t

$$||\widehat{\mathbf{\Theta}}_t - \widehat{\mathbf{\Theta}}_{t-1}||_2 \quad = \quad \sqrt{\lambda_{\max}\{(\widehat{\mathbf{\Theta}}_t - \widehat{\mathbf{\Theta}}_{t-1})(\widehat{\mathbf{\Theta}}_t - \widehat{\mathbf{\Theta}}_{t-1})^\top\}}$$

Copulae over time

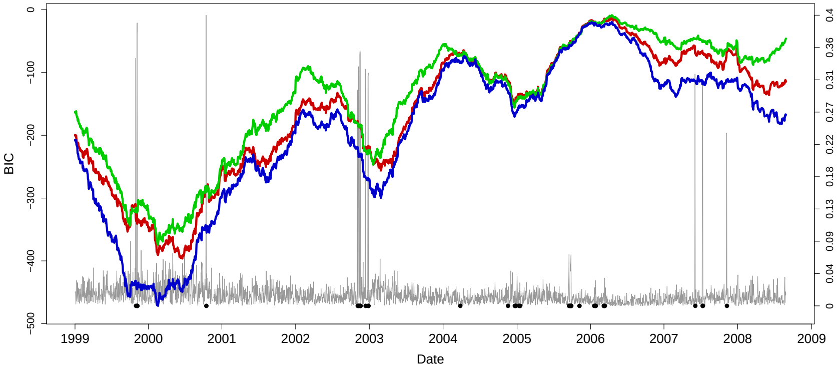


Figure 15: Time-varying HAC: BIC for the AC, Gaussian copula and HAC. Difference Matrix and points of the changes of the structure.

LCP for HAC to real Data

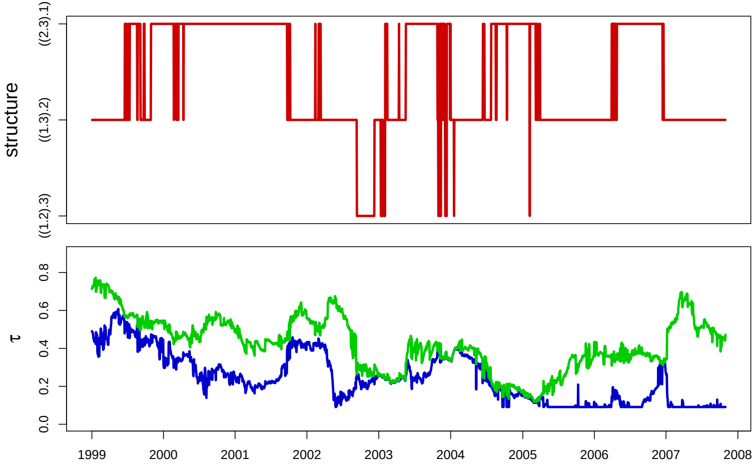


Figure 16: Structure, τ_1 and τ_2 of the HAC on the intervals of homogeneity

LCP for HAC to real Data

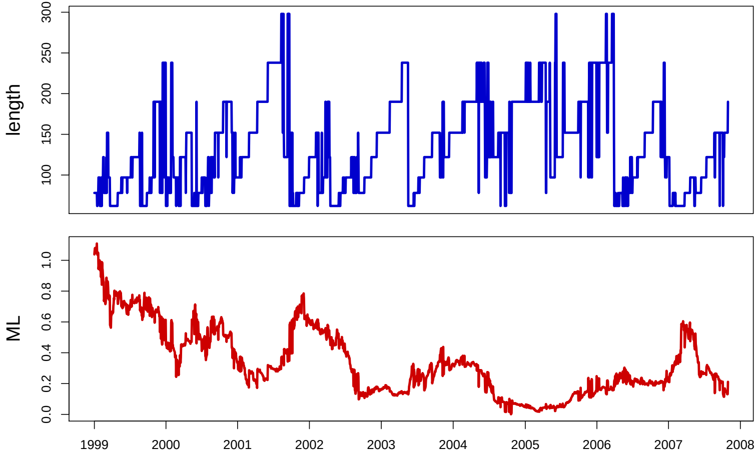


Figure 17: Intervals of homogeneity and ML on these intervals

Data and Copula

daily returns values for Dow Jones (DJ), DAX and NIKKEI

timespan = [4.1.1999; 14.8.2009] ($n = 2771$)

$\mathcal{M} = \{\phi = \exp(-u^{1/\theta})\}$ - Gumbel generator

estimated copula from the whole sample

$$s^* = (\text{DAX DJ})_{2.954}\ \text{NIKKEI}_{1.222}$$

	$\widehat{\mu}_j$	$\widehat{\omega}_j$	$\widehat{\alpha}_j$	$\widehat{\beta}_j$	BL	KS
DAX	6.94e-04	4.17e-06	0.11	0.87	0.23	3.35e-05
	(1.39e-04)	(5.29e-07)	(0.01)	(9.39e-03)		
DJ	5.96e-04	3.09e-06	0.11	0.87	0.02	1.58e-07
	(1.11e-04)	(3.38e-07)	(0.01)	(9.40e-03)		
NIKKEI	5.62e-04	3.01e-06	0.12	0.88	0.78	2.45e-13
	(1.45e-04)	(5.18e-07)	(0.01)	(8.71e-03)		

Table 2: Estimation results univariate time series modelling.

Copulae over time

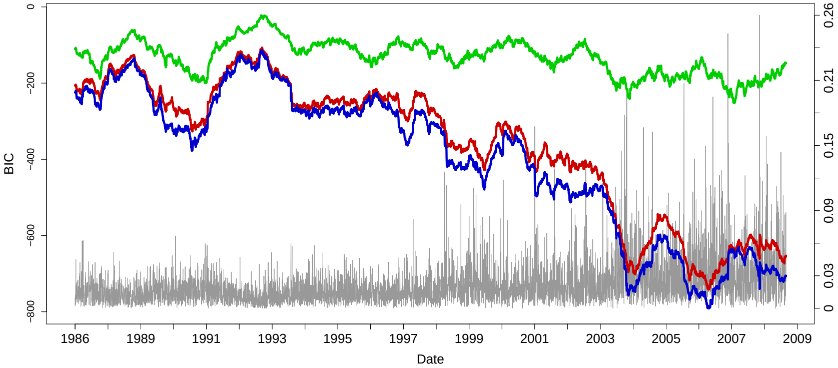


Figure 18: Time-varying HAC: BIC for the AC, Gaussian copula and HAC. Difference Matrix and points of the changes of the structure.

LCP for HAC to real Data

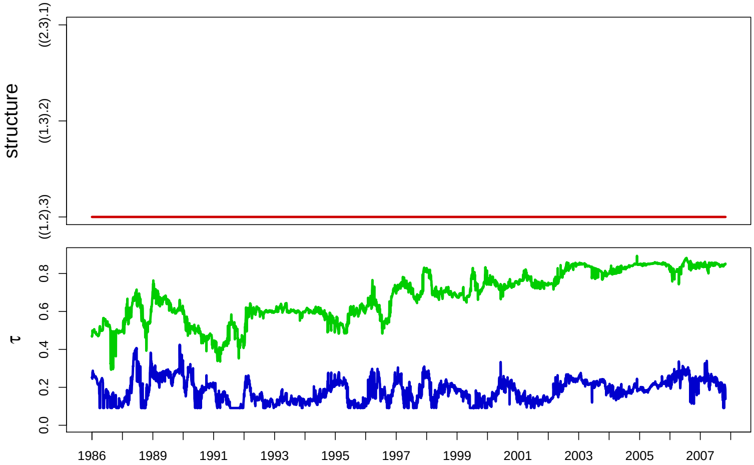


Figure 19: Structure, τ_1 and τ_2 of the HAC on the intervals of homogeneity

LCP for HAC to real Data

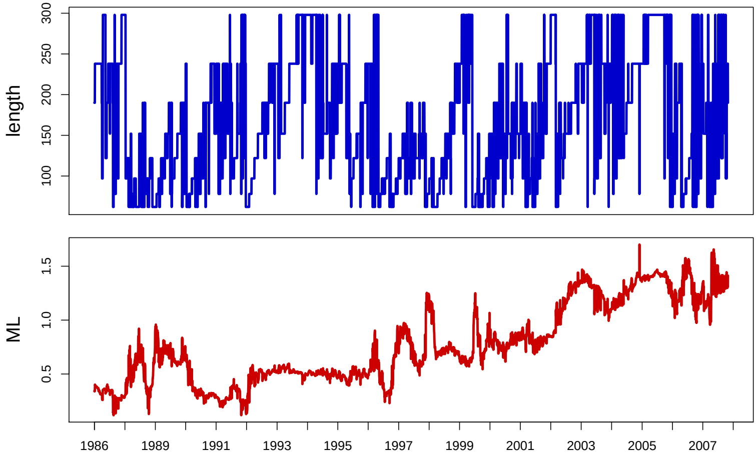


Figure 20: Intervals of homogeneity and ML on these intervals

H. Joe
Multivariate Models and Concept Dependence
Chapman & Hall, 1997

V. Spokoiny
Local Parametric Methods in Nonparametric Estimation
Springer Verlag, 2009

E. Giacomini, W. Härdle and V. Spokoiny
Inhomogeneous Dependence Modeling with Time-Varying Copulae
Journal of Business and Economic Statistics, 27(2), 2009

O. Okhrin and Y. Okhrin and W. Schmid
On the Structure and Estimation of Hierarchical Archimedean Copulas
under revision in *Journal of Econometrics*, 2009